Eur. Phys. J. A **31**, 734–738 (2007)
DOI 10.1140/epja/i2006-10286-2

THE EUROPEAN PHYSICAL JOURNAL A

Special Article – QNP 2006

Superfluidity in many fermion systems: Exact renormalisation group treatment

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Received: 18 December 2006
Published online: 20 March 2007 –

Abstract. The application of the exact renormalisation group to symmetric as well as asymmetric many-fermion systems with a short-range attractive force is studied. Assuming an ansatz for the effective action with effective bosons, describing pairing effects, a set of approximate flow equations for the effective coupling including boson and fermionic fluctuations has been derived. The phase transition to a phase with broken symmetry is found at a critical value of the running scale. The mean-field results are recovered if boson-loop effects are omitted. The calculations with two different forms of the regulator are shown to lead to similar results. We find that, being quite small in the case of the symmetric many-fermion system the corrections to mean-field approximation become more important with increasing mass asymmetry.

PACS. 21.65.+f Nuclear matter – 21.60.-n Nuclear structure models and methods – 68.18.Jk Phase transitions – 73.22.Gk Broken symmetry phases

1 Introduction

There is a growing interest in applying the exact renormalisation group (ERG) formalism to few- and many-body systems [1–3] when the underlying interaction is essentially nonperturbative. Regardless of the details all ERG-based approaches share the same distinctive feature, a successive elimination/suppression of some modes, resulting in effective interaction between the remaining degrees of freedom. One specific way of implementing such a procedure is to eliminate modes by applying a momentum-space blocking transformation with some physically motivated cutoff. The effect of varying a cutoff is described by nonlinear ERG evolution equations, which include the effect of the eliminated modes. In the following we will use the variant of ERG based on the concept of average effective action (AEA) [4]. The corresponding evolution equation can be written in the following general form:

$$\partial_k \Gamma = -\frac{i}{2}\,\mathrm{Tr}\left[(\partial_k R)\,(\Gamma^{(2)} - R)^{-1}\right]. \qquad (1)$$

Here $\Gamma^{(2)}$ is the second functional derivative of the AEA taken with respect to all types of field included in the action, and R is a regulator which should suppress the contributions of states with momenta less than or of the order of the running scale k. To recover the full effective action we require $R(k)$ to vanish as $k \to 0$, in other respects its form is rather arbitrary. The concrete functional form of the regulator has no effect on physical results provided no approximations/truncations were made. By solving the ERG equations one can find a scale dependence of the coupling constants and thus determine the ERG flow. The whole approach is nonperturbative so that some physically motivated assumption about the functional form of the effective action should be made It is well known that the many-fermion system with attractive interaction favours the formation of the correlated fermion pairs leading to the symmetry breaking and related phenomena. Since we expect the appearance of the correlated fermion pairs in a physical ground state, we need to parametrise our effective action in a way that can describe the qualitative change in the physics when this occurs. A natural way to do this is to introduce a boson field whose vacuum expectation value (VEV) describes this correlated pair [5] and study the evolution of this effective degrees of freedom. At the start of the RG evolution, the boson field is not dynamical and is introduced through a Hubbard-Stratonovich transformation of the four-point interaction. As we integrate out more and more of the fermion degrees of freedom by running the cutoff scale k to lower values, we generate dynamical terms in the bosonic effective action. In this paper we treat both symmetric and asymmetric many-fermion systems. The corresponding ansatz for the boson-fermion effective action consists of the kinetic terms for boson and fermions and the interaction term and for two types of fermions it can be written as

$$\Gamma[\mu,k] = \int \mathrm{d}^4x\,(\Gamma_B[\mu,k] + \Gamma_F[\mu,k] + \Gamma_I[k])\,. \qquad (2)$$

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Here $\Gamma_{B(F)}$ is the boson (fermion) part of AEA,

$$\Gamma_B = \phi^\dagger \left(Z_\phi (i\partial_t + \mu_a + \mu_b) + \frac{Z_m}{2m}\nabla^2 \right)\phi - U(\phi, \phi^\dagger), \quad (3)$$

$$\Gamma_F = \sum_{i=a}^{b} \psi_i^\dagger \left(Z_{\psi,i}(i\partial_t + \mu_i) + \frac{Z_{M,i}}{2M_i}\nabla^2 \right)\psi_i\,, \quad (4)$$

and Γ_I is the interaction term,

$$\Gamma_I = -Z_g \left(\frac{i}{2}\,\psi_b^{\mathrm{T}}\sigma_2\psi_a\phi^\dagger - \frac{i}{2}\,\psi_a^\dagger\sigma_2\psi_b^{\dagger\mathrm{T}}\phi \right). \quad (5)$$

M is the reduced mass of the fermion in vacuum and the factor $1/2m$ with $m = M_a + M_b$ in the boson kinetic term is chosen simply to make Z_m dimensionless. The coupling Z_g, the wave function renormalisations factors $Z_{\phi,\psi}$ and the kinetic-mass renormalisations factors $Z_{m,M}$ all run with k, the scale of the regulator. Having in mind the future applications to the crossover from BCS to BEC (where the chemical potential becomes negative) we also let the chemical potentials μ_a and μ_b run, thus keeping the corresponding densities (and Fermi momenta $p_{F,i}$) constant. The bosons are, in principle, coupled to the chemical potentials via a quadratic term in ϕ, but this can be absorbed into the potential by defining $\overline{U} = U - (\mu_1 + \mu_2)Z_\phi\phi^\dagger\phi$. The evolution equations include running of chemical potentials, effective potential and all couplings $(Z_\phi, Z_m, Z_{M,i}, Z_{\psi,i}, Z_g)$. However, in this paper we allow to run only Z_ϕ, parameters in the effective potential ($u's$ and ρ_0) and chemical potentials since this is the minimal set needed to include the effective boson dynamics. The system with one type of fermion corresponds to the limit $M_a = M_b$.

We expand the effective potential about its minimum, $\phi^\dagger\phi = \rho_0$, so that the coefficients u_i are defined at $\rho = \rho_0$,

$$\overline{U}(\rho) = u_0 + u_1(\rho-\rho_0) + \frac{1}{2}\,u_2(\rho-\rho_0)^2 + \frac{1}{6}\,u_3(\rho-\rho_0)^3 + \cdots, \quad (6)$$

where we have introduced $\rho = \phi^\dagger\phi$. A similar expansion can be written for the renormalisation factors. The coefficients of the expansion run with the scale. The phase of the system is determined by the coefficient u_1. We start evolution at high scale where the system is in the symmetric phase so that $u_1 > 0$. When the running scale becomes comparable with the pairing scale (close to average Fermi momentum) the system undergoes the phase transition to the phase with broken symmetry, energy gap etc. The point of the transition corresponds to the scale where $u_1 = 0$. The bosonic excitations in the gapped phase are gapless Goldstone bosons. Note, that in this phase the minimum of the potential will also run with the scale k so that the value $\rho_0(k \to 0)$ determines the physical gap.

The important part of any ERG treatment is the choice of the regulator. Ideally, the physical results should not depend on this choice. However, some sort of truncations and approximations should always be made in real calculations to render the system of the resulting evolution equations solvable so that the convenient choice of the regulator is the question of significant practical importance. In our approach the boson regulator has the structure

$$\mathbf{R}_B = R_B\,\mathrm{diag}(1,1), \quad (7)$$

and the fermion regulator for both types of fermions has the structure

$$\mathbf{R}_{F,i} = \mathrm{sgn}(\epsilon_i(q) - \mu_i)R_{F,i}(q, \mu_i, k)\,\mathrm{diag}(1,-1). \quad (8)$$

Note that this regulator is positive for particle states above the Fermi surface and negative for the hole states below the Fermi surface. The function R_F should suppress the contributions of states with momenta near the Fermi surface, $|q - p_F| \sim k$. Once a large gap has appeared in the fermion spectrum, there are no low-energy fermion excitations and so the fermionic regulator plays little further role. However, while the gap is zero or small, it is crucial that the sign of the regulator matches that of the energy, $q^2/2M - \mu$, and hence it is μ which appears in the sign functions.

The other important part of the ERG approach is fixing the boundary conditions which define the form of the AEA at some initial scale so that at some large starting scale $k = K$ we demand that the Lagrangian be equivalent to a purely fermionic theory with the contact interaction,

$$\mathcal{L}_i = -\frac{1}{4}\,C_0\left(\psi^\dagger\sigma_2\psi^{\dagger\mathrm{T}}\right)\left(\psi^{\mathrm{T}}\sigma_2\psi\right). \quad (9)$$

Here $C_0(K)$ is the strength of the energy-independent term in the effective NN interaction in vacuum. This is evaluated at the scale K, using the same regularisation procedure as we apply in matter. This equation implies that u_1 and g at the this scale are related by

$$C_0(K) = -\frac{g(K)^2}{u_1(K)}\,. \quad (10)$$

Using this boundary condition we can relate the pairing phenomena in a many-body environment with the fermion-fermion interaction in vacuum.

The boson potential $\overline{U}$ is obtained by evaluating the effective action for uniform boson fields. It evolves according to

$$\partial_k\overline{U} = -\frac{1}{\mathcal{V}_4}\,\partial_k\Gamma, \quad (11)$$

where $\mathcal{V}_4$ is the volume of spacetime. Substituting our expansion of $\overline{U}$, eq. (6), on the left-hand side leads to a set of ordinary differential equations for the u_n.

For the boson wave function renormalisation factor, Z_ϕ, we need to consider a time-dependent background field. Taking

$$\phi(x) = \phi_0 + \eta e^{-ip_0 t}, \quad (12)$$

where η is a constant, we can get the evolution of Z_ϕ from

$$\partial_k Z_\phi = \frac{1}{\mathcal{V}_4}\,\frac{\partial}{\partial p_0}\left(\frac{\partial^2}{\partial\eta\partial\eta^\dagger}\,\partial_k\Gamma\right)_{\eta=0}\Bigg|_{p_0=0}. \quad (13)$$

Let us first consider the results of the calculations in the case of symmetric fermion matter. We solve the evolution equations numerically with two types of cutoff. First, we use the smoothed step-function type of regulator (called hereafter as R_{1F}):

$$R_{1F} = \frac{k^2}{2M}\theta_1(q-p_F,k,\sigma); \quad R_{1B} = \frac{k^2}{2m}\theta_1(q,k,\sigma), \quad (14)$$

where

$$\theta_1(q,k,\sigma) = \frac{1}{2\,\mathrm{erf}(1/\sigma)}\left[\mathrm{erf}\left(\frac{q+k}{k\sigma}\right)+\mathrm{erf}\left(\frac{q-k}{k\sigma}\right)\right] \quad (15)$$

with σ being a parameter determining the sharpness of the step. Second, we use the sharp-cutoff function R_{2F} which is somewhat similar to the one suggested in ref. [6] for the pure boson case and chosen in a rather peculiar way to make the calculations as simple as possible

$$R_{2F} = \frac{k^2}{2M}\big[((k+p_\mu)^2-q^2)\theta(p_\mu+k-q) + ((k+p_\mu)^2+q^2-2p_\mu^2)\theta(q-p_\mu+k)\big], \quad (16)$$

$$R_{2B} = \frac{k^2}{2m}(k^2-q^2)\theta(k-q), \quad (17)$$

where $p_\mu = (2M\mu)^{1/2}$. The fermion sharp cutoff consists of two terms which result in the modification of the particle and hole propagators, respectively. The hole term is further modified to suppress the contribution from the surface terms, which may bring in the dangerous dependence of the regulator on the cutoff scale even at the vanishingly small k. As an example, we focus on the parameters relevant to neutron matter: $M = 4.76\,\mathrm{fm}^{-1}$, $p_F = 1.37\,\mathrm{fm}^{-1}$. Let us first discuss the results obtained with the smooth cutoff R_1. We found that the value of the physical gap is practically independent of either the values of the width parameter σ (varied within some range) or the starting scale K provided $K > 5\,\mathrm{fm}^{-1}$. The results of the calculations are shown in fig. 1. At the starting scale the system is in the symmetric phase and remains in this phase until u_1 hits zero at $k_{crit} \simeq 1.2\,\mathrm{fm}^{-1}$ where the artificial second-order phase transition to a broken phase occurs and the energy gap is formed. Already at $k \simeq 0.5$ the running scale has essentially no effect on the gap. It is worth mentioning that we found very small (on the level of 1%) contribution to the gap from the boson loops, due to cancellations between the direct contributions to the running of the gap and indirect ones via u_2. The boson loops play a much more important role in the evolution of u_2 and Z_ϕ. In fact, they drive both couplings to zero at $k \to 0$. We note however, that the effect of the boson loops for the gap may still be more visible if the evolution of the other couplings is included.

The results obtained with the sharp-cutoff regulator are shown in fig. 2. One immediate observation is that the results become starting scale independent as long as $K > 5\,\mathrm{fm}^{-1}$ similarly to the case with the smooth cutoff. However, the artificial phase transition occurs at lower values of the running scale $k \simeq 0.7\,\mathrm{fm}^{-1}$. At approximately $k \simeq 0.2\,\mathrm{fm}^{-1}$ the value of the gap becomes scale independent. One notes that the curves obtained with different regulators and describing the evolution of the gap, being rather different at intermediate scales, approach each other with decreasing scale resulting in very close values for the physical gap. This is an encouraging result taking into account that, although the hypothetical exact results must be independent of the choice of the regulator in practice it is not guaranteed given the assumed ansatz for the effective action and truncations made. The same conclusion also holds for the other quantities. The couplings Z_ϕ and u_2 first grow with scale and then start decreasing eventually coming to zero. The chemical potential begins to decrease at the point of phase transition and becomes scale independent at $k \simeq 0.2\,\mathrm{fm}^{-1}$. However, in this case the numerical values of the chemical potentials obtained with different regulators differ by approximately 20% so that this quantity is more sensitive to the details of effective action and to the trancations made.

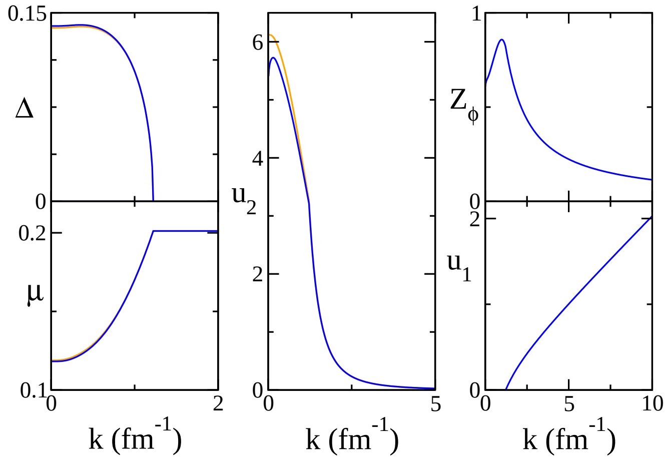

Fig. 1. (Colour on-line) Numerical solutions to the evolution equations for infinite a_0 and $p_F = 1.37\,\mathrm{fm}$, starting from $K = 16\,\mathrm{fm}^{-1}$. We show the evolution of all relevant parameters for the cases of fermion loops only (orange/grey lines), and of bosonic loops with a running Z_ϕ (blue/black lines). All quantities are expressed in appropriate powers of fm^{-1}.

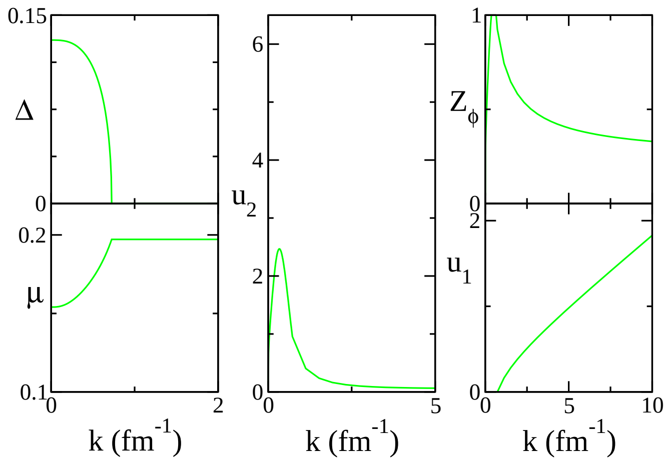

Fig. 2. (Colour on-line) Evolution of the parameters when the sharp cutoff is used.

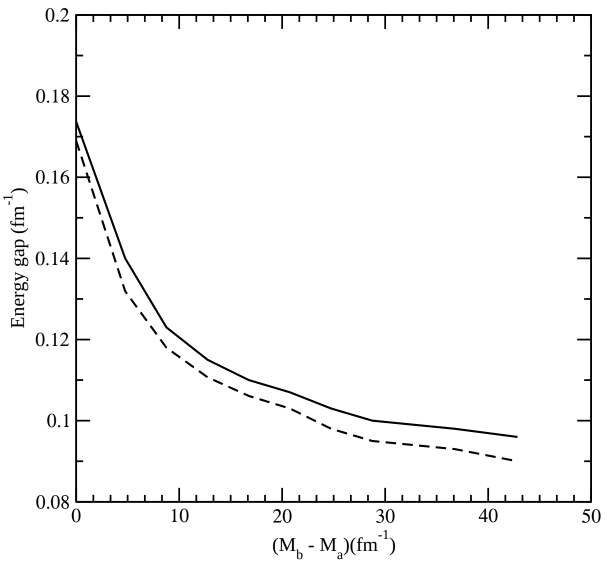

Fig. 3. Evolution of the gap in the MF approach (dashed curve) and with boson loops (solid curve) in the unitary regime $a = -\infty$ as a function of the mass asymmetry.

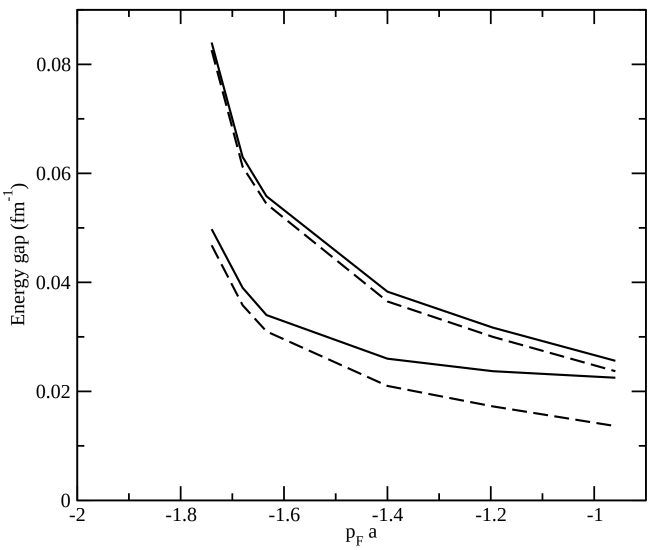

Fig. 4. Evolution of the gap as a function of the parameter $p_F a$. The upper pair of curves corresponds to the calculations with no asymmetry in the MF approach (dashed curve) and with boson loops (solid curve) and the lower pair of curves describes the results of calculations with the maximal asymmetry when $M_b = 10M_a$.

Let us now discuss the results for the many-fermion system with two fermion species. For simplicity we consider the case of the hypothetical "nuclear" matter with short-range attractive interaction between two types of fermions, light and heavy, and study the behaviour of the energy gap as a function of the mass asymmetry. We choose the Fermi momentum to be $p_F = 1.37\,\text{fm}^{-1}$. One notes that the formalism is applicable to any type of a many-body system with two fermion species from quark matter to fermionic atoms so that the hypothetical asymmetrical "nuclear" matter is simply chosen as a study case. We assume that $M_a < M_b$, where M_a is always the mass of the physical nucleon.

First we consider the case of the unitary limit with the infinite scattering length. The results of our calculations for the gap are shown in fig. 3.

We see from this figure that increasing mass asymmetry leads to a decreasing gap that seems to be a natural result. However, the effect of the boson loops is found to be small. We found essentially no effect in the symmetric phase, 2–4% corrections for the value of the gap in the broken phase and even smaller corrections for the chemical potential so that one can conclude that the mean-field approach (MFA) indeed provides the reliable description in the unitary limit for both small and large mass asymmetries. It is worth mentioning that, similar to the above-considered case of the fermion matter with one type of fermion, the boson contributions are more important for the evolution of u_2 where they drive u_2 to zero as $k \to 0$ making the effective potential convex in agreement with the general expectations. This tendency retains in the unitary regime regardless of the mass asymmetry.

We have also considered the behaviour of the gap as a function of the parameter $p_F a$ for the cases of the zero asymmetry $M_a = M_b$ and the maximal asymmetry $M_b = 10M_a$. The results are shown in fig. 4.

One can see from fig. 4 that in the case of zero (or small) asymmetry the corrections stemming from boson loops are small at all values of the parameter $p_F a$ considered here (down to $p_F a = 0.94$). On the contrary, when $M_b = 10M_a$ these corrections, being rather small at $p_F a \geq 2$ become significant ($\sim 40\%$) when the value of $p_F a$ decreases down to $p_F a \sim 1$. We found that at $p_F a \sim 1$ the effect of boson fluctuations becomes $\sim 10\%$ already for $M_b = 5M_a$. One can therefore conclude that the regime of large mass asymmetries, which starts approximately at $M_b > 5M_a$, moderate scattering length and/or the Fermi momenta is the one where the MF description becomes less accurate so that the calculations going beyond the MFA are needed. One might expect that the deviation from the mean-field results could even be stronger in a general case of a large mass asymmetry and the mismatched Fermi surfaces but the detailed conclusion can only be drawn after the actual calculations are performed.

We were not able to follow the evolution of the system at small gap (or small $p_F a$) because of the nonanalyticity of the effective action in this case. This nonanalyticity of the effective action can explicitly be demonstrated in the mean-field approximation. The flow equations can be solved analytically in this case and one can see from the solution, which has a closed-form expression in terms of an associated Legendre function, $P_l^m(y)$ at $k = 0$, that the fermion loops contain a term $\phi^\dagger\phi\log(\phi^\dagger\phi)$. It remains to be seen whether, within the given ansatz, the full solution of the system of the partial differential equations for the effective potential and running couplings is required to trace the evolution of the system in the case of small gaps.

In summary, we have studied the pairing effect for the asymmetric fermion matter with two fermion species as a function of fermion mass asymmetry. We found that regardless of the size of the fermion mass asymmetry the boson loop corrections are small at large enough values of $p_F a$ so that the MFA provides a consistent description of the pairing effect in this case. However, when $p_F a \sim 1$ these corrections become significant at large asymmetries ($M_b > 5M_a$) making the MFA inadequate. In this case it seems to be necessary to go beyond the mean-field description.

There are several ways where this approach can further be developed. In the case of asymmetric systems the next natural step would be to consider the case of the mismatched Fermi surfaces taking into account the possibility of formation of Sarma [7], mixed [8,9] and/or LOFF [10] phases, exploring the importance of the boson loop for the stability of those phases and applying the approach to the real physical systems, for example fermionic atoms. Work in this direction is in progress. The other important extension of this approach would be to take into account running of all couplings of the average effective action, use different types of cutoff function, preferably the smooth one and include both particle-hole channel and long-range forces. The three-body force effects [11], when the correlated pair interact with the unpaired fermion may also be important, especially for nondilute systems.

The author would like to thank Mike Birse, Niels Walet and Judith McGovern for very useful discussions.

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